

Klein-Gordon-Wheeler-DeWitt-Schrödinger Equation

Matej Pavšič

Jožef Stefan Institute, Jamova 39, SI-1000, Ljubljana, Slovenia;

email: matej.pavsic@ijs.si

Abstract

We start from the Einstein-Hilbert action for the gravitational field in the presence of a “point particle” source, and cast the action into the corresponding phase space form. The dynamical variables of such a system satisfy the point particle mass shell constraint, the Hamilton and the momentum constraints of the canonical gravity. In the quantized theory, those constraints become operators that annihilate a state. A state can be represented by a wave functional Ψ that simultaneously satisfies the Klein-Gordon and the Wheeler-DeWitt-Schrödinger equation. The latter equation, besides the term due to gravity, also contains the Schrödinger like term, namely the derivative of Ψ with respect to time, that occurs because of the presence of the point particle. The particle’s time coordinate, X^0 , serves the role of time. Next, we generalize the system to p -branes, and find out that for a quantized spacetime filling brane there occurs an effective cosmological constant, proportional to the expectation value of the brane’s momentum, a degree of freedom that has two discrete values only, a positive and a negative one. This mechanism could be an explanation for the small cosmological constant that drives the accelerated expansion of the universe.

1 Introduction

The meaning of time in quantum gravity is still a matter of debate (for a recent review see [1]). A possible resolution of this problem is to consider matter degrees of freedom from which, upon quantization, one can obtain the derivative of the wave functional with respect to a time variable [2] in the Wheeler-DeWitt equation [3]. The idea is to introduce a reference fluid [4], which enables the identification of spacetime points and the occurrence of a time variable. Instead of a fluid, one can consider a model with one point particle only [2]. In this letter we will further explore and adapt that model. We start with the Einstein-Hilbert action for gravity in the presence of a “point particle” source that is in fact an extended object, like a ball, whose center of mass worldline satisfies the equations of motion for a point particle. Then we cast the action into the phase space form that involves the set of Lagrange multipliers: α , the einbein on the particle’s worldline, N , the lapse and N^i , the shift functions that occur in the ADM decomposition [5] of the spacetime metric tensor

$g_{\mu\nu}$. Variation of the action with respect to α , N and N^i gives the mass shell constraint, the Hamilton and the momentum constraint. In the quantized theory, such system is described by a wave functional $\Psi[X^\mu, q_{ij}]$ that satisfies the Klein-Gordon and the Wheeler-DeWitt-Schrödinger equation. The latter equation contains, besides the usual Wheeler-DeWitt terms due to gravity, also the term $\delta^3(\mathbf{x} - \mathbf{X}) i\partial\Psi/\partial X^0$ due to the point particle. In addition, the wave functional also satisfies the quantum momentum constraint that contains the term $\delta^3(\mathbf{x} - \mathbf{X}) i\partial\Psi/\partial X^i$, $i = 1, 2, 3$.

In distinction to Rovelli, we do not introduce here an extra, the so called “clock dynamical variable”, associated with the particle. In our approach we use the time component $x^0 = X^0(\tau)$ of the worldline parametric equation $x^\mu = X^\mu(\tau)$, and fix the parameter τ by requiring $X^0(\tau) = \tau$. It turns out that the particle coordinate X^0 serves as evolution parameter, just like in field theories. That X^0 , which is not the particle dynamical degree of freedom, serves as time is in agreement with the well known fact that time in quantum mechanics is not a dynamical degree of freedom, but merely a parameter. According to this line of reasoning, we do not need to worry how to find a dynamical variable with the role of time. It comes out that $t \equiv x^0 = X^0$, i.e., the quantity that in special and general relativity we anyway call “time”, is indeed time, since it can serve as an evolution parameter. This happens, if we do not consider gravity in empty space, but gravity in the presence of a point particle for which it is no problem to identify X^0 as time.

Next, we consider the gravity in the presence of many particles, and finally in the presence of a p -brane. Then, instead of one time, we have the many fingered time $X^0(\sigma^a)$, $a = 1, 2, \dots, p$. The wave functional for the brane satisfies, besides the Wheeler-DeWitt-Schrödinger equation, also the quantum p -brane constraints that replace the Klein-Gordon equation. We explore a special case of a spacetime filling brane, and obtain the positive or negative cosmological constant that depends on the sign of the brane momentum p_0 . The latter momentum, because of the p -brane constraint, has two discrete values only, $p_0 = +\mu_B\sqrt{q}$ and $p_0 = -\mu_B\sqrt{q}$, where μ_B is the brane tension and q the determinant of the 3-space metric. The quantized theory then gives an expectation value $\langle\hat{p}_0\rangle$ for the state that is a superposition of the eigenstates with positive and negative p_0 . The effective cosmological constant, proportional to $\langle\hat{p}_0\rangle$, has thus a continuous range of possible values, including the one that fits the observed accelerated expansion of the universe. At the end we discuss the possibility that a 3-brane in a higher dimensional bulk space is our world—a “brane world.”

2 The Einstein-Hilbert action with a “point particle” matter term and its quantization

Let us consider the Einstein-Hilbert action for the gravitational field $g_{\mu\nu}(x)$, $\mu, \nu = 0, 1, 2, 3$, in the presence of a “point particle” source, described by variables $X^\mu(\tau)$:

$$I[X^\mu, g_{\mu\nu}] = m \int d\tau (\dot{X}^\mu \dot{X}^\nu g_{\mu\nu})^{1/2} + \kappa \int d^4x \sqrt{-g} R \quad (1)$$

where $\kappa \equiv 1/(16\pi G)$. It is well known that the Einstein equations with a point like source have no solution, because a solution in the vacuum around a source is the black hole with a horizon, the black hole singularity being spacelike and cannot hence be interpreted as a point particle worldline. However, for the sake of completeness, let me mention that alternative views can be found in the literature [6]. Leaving such intricacies aside, we can nevertheless use the action (1) as an approximation to a realistic physical situation in which instead of a point particle we have an extended source, described by $X^\mu(\tau, \sigma^a)$, with $X^\mu(\tau)$ being the center of mass coordinates. In particular, if the particle is a ball, then the parameters are $\sigma^a = (R, \theta, \phi)$, $0 < R < R_0$, $0 < \theta < \pi$, $0 < \phi < 2\pi$, where R_0 is greater than the Schwarzschild radius.

Let us now consider the ADM split of spacetime, $M_{1,3} = \mathbb{R} \times \mathbb{R}^{0,3}$. Then the 4D metric can be decomposed as

$$g_{\mu\nu} = \begin{pmatrix} N^2 - N^i N_i & -N_i \\ -N_j & -q_{ij} \end{pmatrix} \quad (2)$$

where $N = \sqrt{1/g^{00}}$ and $N_i = -g_{0i}$, $i = 1, 2, 3$, are the laps and shift functions. The inverse metric is

$$g^{\mu\nu} = \begin{pmatrix} 1/N^2 & -N^i/N^2 \\ -N^j/N^2 & N^i N^j/N^2 - q^{ij} \end{pmatrix} \quad (3)$$

Here q^{ij} is the inverse of q_{ij} and $N^i = q^{ij} N_j$.

The gravitational part of the action (1) can be cast, by using Ref. [3], into the phase space form [7]:

$$I_G[q_{ij}, \pi^{ij}, N, N_i] = \int dt d^3x [\pi^{ij} \dot{q}_{ij} - N \mathcal{H}(q_{ij}, \pi^{ij}) - N_i \mathcal{H}^i(q_{ij}, \pi^{ij})], \quad (4)$$

where

$$\mathcal{H} = -\frac{1}{\kappa} G_{ij\,kl} \pi^{ij} \pi^{kl} + \kappa \sqrt{q} R^{(3)} \quad (5)$$

$$\mathcal{H}^i = -2D_j \pi^{ij} \quad (6)$$

and

$$G_{ij\,kl} = \frac{1}{2\sqrt{q}} (q_{ik} q_{jl} + q_{il} q_{jk} - q_{ij} q_{kl}) \quad (7)$$

is the Wheeler-DeWitt metric. If we vary the gravitational action with respect to N , N_i , we obtain the constraints

$$\mathcal{H} = 0 \quad (8)$$

$$\mathcal{H}^i = 0 \quad (9)$$

Variation with respect to π^{ij} gives the relation

$$\pi^{ij} = \kappa\sqrt{q}(K^{ij} - Kq^{ij}) \quad (10)$$

where

$$K_{ij} = \frac{1}{2N}(\mathcal{D}_i N_j + \mathcal{D}_j N_i - \dot{q}_{ij}) \quad (11)$$

The matter part of the action can also be cast into the phase space form:

$$I_m[X^\mu, p_\mu, \alpha] = \int d\tau \left[p_\mu \dot{X}^\mu - \frac{\alpha}{2}(g_{\mu\nu} p^\mu p^\nu - m^2) \right] \quad (12)$$

To cast the matter part into a form comparable to the gravitational part of the action, we insert the integration over $\delta^4(x - X(\tau))d^4x$, which gives identity. In both parts of the action, I_m and I_G , now stands the integration over d^4x . We identify $x^0 \equiv t$.

Splitting the metric according to (2), we have

$$\begin{aligned} I_m[X^\mu, p_\mu, \alpha, N, N_i, q_{ij}] \\ = \int d\tau \left(p_\mu \dot{X}^\mu - \frac{\alpha}{2} [N^2(p^0)^2 - q_{ij}(p^i + N^i p^0)(p^j + N^j p^0) - m^2] \right) \end{aligned} \quad (13)$$

Varying the total action

$$I = I_G + I_m \quad (14)$$

with respect to α , N and N^i we obtain the following constraints¹:

$$\delta\alpha : \quad N^2(p^0)^2 - q_{ij}(p^i + N^i p^0)(p^j + N^j p^0) - m^2 = 0 \quad (15)$$

$$\begin{aligned} \delta N : \quad \mathcal{H} &= \int d\tau \alpha N \delta^4(x - X(\tau))(p^0)^2 \\ &= -\delta^3(\mathbf{x} - \mathbf{X}) N p^0 \end{aligned} \quad (16)$$

$$\begin{aligned} \delta N^i : \quad \mathcal{H}_i &= \int d\tau \alpha N \delta^4(x - X(\tau)) q_{ij} (p^j + N^j p^0) p^0 \\ &= \delta^3(\mathbf{x} - \mathbf{X}) q_{ij} (p^j + N^j p^0) \end{aligned} \quad (17)$$

¹Since a realistic source is extended, e.g. like a “ball”, $\int d\tau \delta^4(x - X(\tau))$ should be considered as an approximation to $\int d\tau d^3\sigma \delta^4(x - X(\tau, \sigma^a))$, so that, e.g., the constraint $\mathcal{H} = -\delta^3(\mathbf{x} - \mathbf{X}) N p^0$ is an approximation to $\mathcal{H} = -\int d^3\sigma \delta^3(\mathbf{x} - \mathbf{X}(\sigma^a)) N p^0$.

where \mathcal{H} and $\mathcal{H}_i = q_{ij}\mathcal{H}^j$ are given in Eqs. (5),(6). Eq. (15), of course, is nothing but the ADM splitting of the mass shell constrain

$$g^{\mu\nu}p_\mu p_\nu - m^2 = 0 \quad (18)$$

In Eqs. (16),(17) we have performed the integration over τ , and used the equation $p^\mu = \dot{X}^\mu/\alpha$, that results from varying the action (12) with respect to p^μ .

Let us now use the relations $p^\mu = g^{\mu\nu}p_\nu$ and $p_\mu = g_{\mu\nu}p^\nu$ with the metrics (2),(3), and rewrite (16),(17) into the form with covariant components of momenta p_0, p_i :

$$\mathcal{H} = -\delta^3(\mathbf{x} - \mathbf{X})\frac{1}{N}(p_0 - N^i p_i) \quad (19)$$

$$\mathcal{H}_i = -\delta^3(\mathbf{x} - \mathbf{X})p_i \quad (20)$$

In a quantized theory, the constraints (15)–(17) become operator equations acting on a state vector. In the Schrödinger representation, in which X^μ and $q_{ij}(x)$ are diagonal, the momentum operators are $\hat{p}_\mu = -i\partial/\partial X^\mu$ and $\hat{\pi}^{ij} = -i\delta/\delta q^{ij}$. More precisely, momentum operators have to satisfy the condition of hermiticity, therefore the latter definitions are not quite correct in curved spaces, and have to be suitably modified. For instance, a possible definition [8] that renders \hat{p}_μ hermitian, and also helps to resolve the factor ordering ambiguity, is $\hat{p}_\mu = -i[\partial_\mu + (-g)^{-1/4}\partial_\mu(-g)^{1/4}]$. An alternative procedure was proposed in Ref. [9]. Analogous holds for $\hat{\pi}^{ij}$.

Choosing a gauge in which $N = 1$, $N^i = 0$, we have

$$(g^{\mu\nu}(X)\hat{p}_\mu\hat{p}_\nu - m^2)\Psi = 0 \quad (21)$$

$$\hat{\mathcal{H}}\Psi = \delta^3(\mathbf{x} - \mathbf{X})i\frac{\partial\Psi}{\partial T} \quad (22)$$

$$\hat{\mathcal{H}}_i\Psi = \delta^3(\mathbf{x} - \mathbf{X})i\frac{\partial\Psi}{\partial X^i} \quad (23)$$

A state vector is represented by $\Psi[T, X^i, q_{ij}(\mathbf{x})]$ that depends on the time parameter $T \equiv X^0$, the particle center of mass coordinates² X^i , and the 3-metric $q_{ij}(\mathbf{x})$. In other words, Ψ is a function of T , X^i , and a functional of $q_{ij}(\mathbf{x})$. It satisfies simultaneously the Klein-Gordon equation (21), the Wheeler-DeWitt-Schrödinger like equation (22), and the quantum momentum constraint (23) in the presence of a point particle source. However, Eq. (22) is not the Schrödinger evolution equation; it is a constraint that

²We assume that the coupled system actually describes an extended particle whose center of mass coordinates are X^μ . This system can be envisaged to describe, e.g., the neutron that certainly is extended, and yet only its center of mass coordinates can be considered in the wave function. If we wish to use the above coupled system for description of electron and other fundamental particles, one has to assume that they are as well extended beyond their Schwarzschild radii. Otherwise those “particles” would be black holes. Since the underlying physical system whose description we have in mind, is in fact extended, it has classical solutions.

has to be satisfied at every point \mathbf{x} . Since \mathbf{x} runs over the 3-manifold, we have in fact an infinite set of constraints.

Usually, for a quantum description of gravity in the presence of matter, one does not take the matter action in the form (12). Instead, one takes [10] for I_m an action for, e.g., a scalar or spinor field, and then attempts to quantize the total action following the established procedure of quantum field theory. Here I have pointed out that we can nevertheless start from the point particle action (12) together with the corresponding gravitational action (4). After quantization, we arrive at the Klein-Gordon equation (21) and the equations (22),(23) that are the Wheeler-DeWitt equation, and the momentum constraint, with the terms due to the presence of point particle source.

The presence of the δ -distribution can be avoided, if we perform the Fourier transform. The classical constraints (19),(20), with $N = 1$, $N^i = 0$, then become

$$H(\mathbf{k}) = -e^{i\mathbf{k}\mathbf{X}} p_0 \quad (24)$$

$$H_i(\mathbf{k}) = -e^{i\mathbf{k}\mathbf{X}} p_i \quad (25)$$

where

$$H(\mathbf{k}) = \int d^3x e^{i\mathbf{k}\mathbf{x}} \mathcal{H}, \quad H_i(\mathbf{k}) = \int d^3x e^{i\mathbf{k}\mathbf{x}} \mathcal{H}_i \quad (26)$$

and $\mathbf{X} \equiv X^i$, $i = 1, 2, 3$, is the particle's position at fixed time T . Notice that $\mathbf{k} \equiv k^i$ are the Fourier partners of the spacetime coordinates \mathbf{x} , not of the particle position \mathbf{X} .

The quantum constraint are³

$$\int d^3x e^{i\mathbf{k}(\mathbf{x}-\mathbf{X})} \left(\frac{1}{\kappa} G_{ij\,k\ell} \frac{\delta^2}{\delta q_{ij} \delta q_{k\ell}} + \kappa \sqrt{q} R^{(3)} \right) \Psi = i \frac{\partial \Psi}{\partial T} \quad (27)$$

$$- 2 \int d^3x e^{i\mathbf{k}(\mathbf{x}-\mathbf{X})} q_{i\ell} D_j \left(-i \frac{\delta}{\delta q_{j\ell}} \right) \Psi = i \frac{\partial \Psi}{\partial X^i} \quad (28)$$

The above \mathbf{k} -dependent set of constraints (27),(28) replaces the set of constraints (22),(23). For a fixed \mathbf{k} , Eq. (27) has the form of the Schrödinger equation, with the Hamilton operator that contains the functional derivatives $-i\delta/\delta q_{ij}$. In the Hamiltonian we have the integration over \mathbf{x} , just as in the Hamiltonians of the usual field theories.

The zero mode Schrödinger equation, for $\mathbf{k} = \mathbf{0}$, is

$$\int d^3x \left(\frac{1}{\kappa} G_{ij\,k\ell} \frac{\delta^2}{\delta q_{ij} \delta q_{k\ell}} + \kappa \sqrt{q} R^{(3)} \right) \Psi = i \frac{\partial \Psi}{\partial T} \quad (29)$$

³We are not interested here in the issues of hermiticity and factor ordering, therefore the expressions with $-i\delta/\delta q_{ij}$ have symbolical meaning only. In actual calculation one has to take suitable hermitian operators, and choose a factor ordering.

A solution $\Psi = \Psi_0$ of Eq. (29) is an approximate solution of our dynamical system. Correction terms to Ψ_0 come from the contributions from the higher modes, $\mathbf{k} \neq \mathbf{0}$, in Eq. (27). Bear in mind that the momentum constraints are a consequence [11] of the conservation of the Hamiltonian constraint with respect to x^0 .

3 Generalization to many particle and extended sources

If instead of one, there are many particle sources, then the matter part of the action, I_m , consists of the sum over single particle sources:

$$I_m[X_n^\mu, p_{n\mu}, \alpha_n] = \sum_n \int d\tau_n \left[p_{n\mu} \dot{X}_n^\mu - \frac{\alpha_n}{2} (g^{\mu\nu} p_{n\mu} p_{n\nu} - m_n^2) \right] \quad (30)$$

As a consequence, in the constraints (19),(20), instead of a single δ -distribution, we have a sum. The quantum equations (21)–(23) become

$$(g^{\mu\nu}(X_n) \hat{p}_{n\mu} \hat{p}_{n\nu} - m_n^2) \Psi = 0 \quad (31)$$

$$\hat{\mathcal{H}}\Psi = \sum_n \delta^3(\mathbf{x} - \mathbf{X}_n) i \frac{\partial \Psi}{\partial T_n} \quad (32)$$

$$\hat{\mathcal{H}}_i \Psi = \sum_n \delta^3(\mathbf{x} - \mathbf{X}_n) i \frac{\partial \Psi}{\partial X_n^i} \quad (33)$$

The Wheeler-DeWitt equation thus becomes a multi fingered time equation. Its Fourier transform is

$$\int d^3\mathbf{x} e^{i\mathbf{k}\mathbf{x}} \mathcal{H}\Psi = i \sum_n e^{i\mathbf{k}\mathbf{X}_n} \frac{\partial \Psi}{\partial T_n} \quad (34)$$

We can single out one particle, denote its time and spatial coordinates as T and \mathbf{X} , respectively, and rewrite Eq. (34) according to

$$\int d^3\mathbf{x} e^{i\mathbf{k}(\mathbf{x}-\mathbf{X})} \mathcal{H}\Psi = i \sum_{n=1}^{N-1} e^{i\mathbf{k}(\mathbf{X}_n-\mathbf{X})} \frac{\partial \Psi}{\partial T_n} + i \frac{\partial \Psi}{\partial T} \quad (35)$$

One particle, in the above case the N^{th} one, was singled out and chosen as a clock that measures a time T . The name ‘particle’ in the quantum equation (35) should be taken with caution. In fact we have a system of many gravitationally interacting particles, described by $\Psi[T_n, \mathbf{X}_n, q_{ij}(\mathbf{x})]$, $n = 1, 2, \dots, N$, and if particles are indistinguishable, one cannot say which particle is at which position. What we have singled out was in fact one of the parameters T_n , namely $T_N \equiv T$.

Instead of a system of many particles, we can consider an extended source, for instance, a p -brane. Then we have [12]–[16]:

$$I_m[X^\mu, p_\mu, \alpha, \alpha^a] = \int d\tau d^p\sigma \left[p_\mu \dot{X}^\mu - \frac{\alpha}{2\mu_B \sqrt{|\bar{f}|}} (g^{\mu\nu} p_\mu p_\nu + \mu_B^2 \bar{f}) - \alpha^a \partial_a X^\mu p_\mu \right] \quad (36)$$

Here μ_B is the brane tension, $\tau, \sigma^a, a = 1, 2, \dots, p$, the brane time like and space like parameters, α, α^a , Lagrange multipliers, $i, j = 1, 2, \dots, D-1$, the spatial indices of the D -dimensional spacetime in which the brane is embedded, and $\bar{f} \equiv \det \bar{f}_{ab}$ the determinant of the induced metric $\bar{f}_{ab} \equiv \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$.

If we vary the action (36) with respect to α, α^a , we obtain the p -brane constraints [13, 15]:

$$g^{\mu\nu} p_\mu p_\nu + \mu_B^2 \bar{f} = 0, \quad \partial_a X^\mu p_\mu = 0 \quad (37)$$

and if we vary the total action $I_g + I_m$ with respect to N, N^i , we obtain the constraints

$$\mathcal{H} = - \int d^p\sigma p_0 \delta^{D-1}(\mathbf{x} - \mathbf{X}(\sigma)) \quad (38)$$

$$\mathcal{H}_i = - \int d^p\sigma p_i \delta^{D-1}(\mathbf{x} - \mathbf{X}(\sigma)) \quad (39)$$

Varying the action (36) with respect to p_μ , we obtain the relation between momenta and velocities:

$$p_\mu = \frac{\mu_B \dot{X}_\mu \sqrt{-\bar{f}}}{\alpha} \quad (40)$$

Squaring the latter equation and combining it with Eq. (37), we obtain $\alpha^2 = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu}$.

As in the case of a point particle, there are difficulties with classical equations of motion of the branes coupled to the gravitational field in all cases except with the appropriate codimension [14]. But again, instead of infinitely thin branes, we can consider thick branes, and interpret the distribution $\delta^{D-1}(\mathbf{x} - \mathbf{X}(\sigma))$ as an approximation of the corresponding distribution for the thick brane.

In the quantized theory, we replace⁴ $p_\mu(\sigma) \rightarrow -i\delta/\delta X^\mu(\sigma)$. Instead of Eqs. (31)–(33), we have

$$\left(-g^{\mu\nu} \frac{\delta^2}{\delta X^\mu(\sigma) \delta X^\nu(\sigma)} + \mu_B^2 \bar{f} \right) \Psi = 0, \quad \partial_a X^\mu \frac{\delta \Psi}{\delta X^\mu(\sigma)} = 0 \quad (41)$$

$$\hat{\mathcal{H}} \Psi = i \int d^p\sigma \delta^{D-1}(\mathbf{x} - \mathbf{X}(\sigma)) \frac{\delta \Psi}{\delta T(\sigma)} \quad (42)$$

$$\hat{\mathcal{H}}_i \Psi = i \int d^p\sigma \delta^{D-1}(\mathbf{x} - \mathbf{X}(\sigma)) \frac{\delta \Psi}{\delta X^i(\sigma)} \quad (43)$$

⁴See footnote 1 and text after Eq. (20)

where $T(\sigma) \equiv X^0(\sigma)$, $\mathbf{X}(\sigma) \equiv X^i(\sigma)$, and $\sigma \equiv \sigma^a$, $a = 1, 2, \dots, p$. In general, $\Psi = \Psi[X^\mu(\sigma), q_{ij}(\mathbf{x})] \equiv \Psi[T(\sigma), \mathbf{X}(\sigma), q_{ij}(\mathbf{x})]$. Since now we have a spacetime of arbitrary dimension D , the definitions (5),(6) of \mathcal{H} and \mathcal{H}_i have to be modified accordingly: $R^{(3)}$ should be replaced by $R^{(D-1)}$, and the Wheeler-DeWitt metric is now $G_{ijkl} = (1/(2\sqrt{q}))[q_{ik}q_{jl} + q_{il}q_{jk} - (2/(D-2))q_{ij}q_{kl}]$.

Of particular interest are the following special cases:

(i) *The spacetime filling brane.* Then $p = D - 1$, and $i = a = 1, 2, \dots, D - 1$. One can choose a parametrization of σ^a such that $X^i(\sigma) = \delta^i_a \sigma^a$. Then we have $\partial_a X^i(\sigma) = \delta^i_a$. The second constraint (41) then reads $\delta\Psi/\delta X^i = 0$. This means that $\Psi = \Psi[T(\sigma), q_{ij}(\mathbf{x})]$, i.e., it does not depend on spatial functions $X^i(\sigma)$. Therefore, the first constraint (41) retains the T -derivatives only:

$$\left(-g^{00}\frac{\delta^2}{\delta T(\sigma)\delta T(\sigma)} + \mu_B^2 \bar{f}\right)\Psi = 0 \quad (44)$$

If we now assume $\partial_a T(\sigma) = 0$, the functional derivative can be replaced by the partial derivative according to the relation $\delta/\delta T(\sigma) \rightarrow \sqrt{-\bar{f}} \partial/\partial T$. Then, instead of (44), we have

$$-\frac{1}{N}\frac{\partial}{\partial T}\left(\frac{1}{N}\frac{\partial\Psi}{\partial T}\right) - \mu_B^2\Psi = 0 \quad (45)$$

The factor ordering has been chosen in order to achieve covariance in the one dimensional space comprised of T . The constraints (42),(43) become

$$\hat{\mathcal{H}}\Psi = \sqrt{q} i \frac{\partial\Psi}{\partial T}, \quad \hat{\mathcal{H}}_i\Psi = 0 \quad (46)$$

where we have taken into account the relation $\bar{f}_{ab} \equiv \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} = g_{ab} = -q_{ab} = -q_{ij}$, and $\bar{f} = -q \equiv -\det q_{ij}$.

Eq. (45), in which we take $N = 1$, implies that a general solution $\Psi[T, q_{ij}(\mathbf{x})]$ is a superposition of particular solutions $\Psi_+ = e^{+i\mu_B T} \psi[q_{ij}(\mathbf{x})]$ and $\Psi_- = e^{-i\mu_B T} \psi[q_{ij}(\mathbf{x})]$ that are eigenfunctions of the operator $\hat{p}_0/\sqrt{q} = -i\partial/\partial T$ with eigenvalues $\pm\mu_B$. For such particular solutions, the quantum Hamilton constraint equation becomes

$$\hat{\mathcal{H}}\Psi_\pm = \mp\sqrt{q}\mu_B\Psi_\pm \quad (47)$$

The expectation value of the operator \hat{p}_0/\sqrt{q} in a superposition state $\Psi = \alpha\Psi_+ + \beta\Psi_-$ is $\langle\hat{p}_0/\sqrt{q}\rangle = (|\alpha|^2 - |\beta|^2)\mu_B$, where $|\alpha|^2 + |\beta|^2 = 1$.

That there must be plus or minus sign in Eq. (47), can be seen already at the classical level. For a spacetime filling brane, the Hamilton constraint (38) becomes $\mathcal{H} = -p_0$. From Eq. (40) we have $p_0 = \mu_B \dot{X}^\mu \sqrt{q}/\alpha$, where $\alpha = \sqrt{\dot{X}^0 \dot{X}^0 g_{00}} = |\dot{X}^0| \sqrt{g_{00}} = |\dot{X}^0|$ is taken to be a positive quantity. In the last step we have used $g_{00} = N^2 - N^i N_j$ and set $N = 1$, $N^i = 0$. Thus we obtain that $p_0 = \mu_B \sqrt{q} \dot{X}^0 / |\dot{X}^0| =$

$\pm\mu_B\sqrt{q}$, depending on whether \dot{X}^0 is positive or negative. In other words, the sign of p_0 depends on whether the spacetime filling brane moves forward or backwards in time. Despite that the momentum p_0 of a spacetime filling brane, because of the constraint (37), which now reads $p_0^2 = \mu_b^2 q$, is not a continuous dynamical degree of freedom, there still remains a freedom for p_0 to be either positive or negative, more precisely, to be $p_0 = \mu_b \sqrt{q}$ or $p_0 = -\mu_b \sqrt{q}$.

It is illustrative to look at the situation from another angle. The p -brane phase space action (36) is equivalent to the minimal surface action

$$I_m[X^\mu(\xi)] = \mu_B \int d^{p+1}\xi \sqrt{-\det \partial_A X^\mu \partial_B X^\nu g_{\mu\nu}} \quad (48)$$

where $\xi^A = (\tau, \sigma^a)$. Performing the ADM split on the brane's world manifold, this can be written as [15]

$$I_m[X^0, X^i] = \mu_B \int d\tau d^p\sigma \sqrt{\dot{X}^\mu \dot{X}^\nu g_{\mu\nu}} \sqrt{-\bar{f}} \quad (49)$$

For a spacetime filling brane we have $X^i = \delta^i_a \sigma^a$, $D = p + 1$, and the latter action becomes

$$\begin{aligned} I_m[X^0] &= \mu_B \int d\tau d^p\sigma \sqrt{\dot{X}^0 \dot{X}^0 g_{00}} \sqrt{q} = \mu_B \int dX^0 d^p\mathbf{X} \frac{|\dot{X}^0|}{\dot{X}^0} N \sqrt{q} \\ &= \pm \mu_B \int dx^0 d^p\mathbf{x} N \sqrt{q} \end{aligned} \quad (50)$$

where we have used $g_{00} = N^2$, $N^i = 0$, $\bar{f} = -q$, and identified X^μ with x^μ . The variation of the latter action with respect to N gives

$$\frac{\delta I_m}{\delta N} = \pm \sqrt{q} \mu_B \quad (51)$$

The function $X^0(\tau)$, where τ is a monotonically increasing parameter, has no physical meaning; it depends on choice of coordinates. Therefore, the derivative \dot{X}^0 has no physical meaning as well. However, there exist two possibilities. One possibility is that $X^0(\tau)$ increases with τ . Another possibility is that $X^0(\tau)$ decreases with τ . We assume that these two different possibilities correspond to physically different situations, because they lead, respectively, to the positive and negative cosmological constant. They provide an explanation for the positive or negative sign in Eq. (47).

We can look at the situation even more directly. Since in the case of a spacetime filling brane its world volume fills the embedding spacetime, we can choose coordinates in the action (48) so that $X^\mu(\xi^A) = \delta^\mu_A \xi^A = \xi^\mu$. Bear in mind that now $\mu = 0, 1, 2, \dots, D - 1$ and $A = 0, 1, 2, \dots, p = D - 1$. By such choice of coordinates, we obtain $I_m = \mu_B \int d^D x \sqrt{-g}$. But we may as well choose $X^0(\xi^A) = -\tau$, where $\tau \equiv$

ξ^0 , which means that X^0 increases in the opposite direction than τ does, and so, figuratively speaking, our brane “moves backward in time”. Then we obtain $I_m = -\mu_B \int d^4x \sqrt{-g}$. This corresponds to the term with the cosmological constant $\Lambda = \pm 16\pi G \mu_B = 16\pi G p_0/\sqrt{q}$. The spacetime filling brane is thus responsible for the cosmological constant, which can be positive or negative. In the quantized theory, a generic state is a superposition of those two possibilities, $\Psi = \alpha\Psi_+ + \beta\Psi_-$, where the eigenstates Ψ_{\pm} simultaneously satisfy Eq. (45) and (46). We have thus verified that, in the case of the spacetime filling brane the system of equations (41)–(43) has a consistent solution. For the expectation value of the cosmological constant in the superposition state Ψ we obtain

$$\langle \hat{\Lambda} \rangle = 16\pi G \langle \frac{\hat{p}_0}{\sqrt{q}} \rangle = (|\alpha|^2 - |\beta|^2) 16\pi G \mu_B \quad (52)$$

It can be any value between $16\pi G \mu_B$ and $-16\pi G \mu_B$, including zero or a small value that fits the accelerated expansion of the universe. That a spacetime filling brane gives the cosmological constant was considered by Bandos [17], but he took into account one sign only.

(ii) *The brane is a brane world.*

Another possibility is that a 3-brane, embedded in a higher dimensional spacetime (bulk), is our observable world (“brane world”) [18]. Then everything that directly counts for us as observers are points on the brane. It does not matter that points outside the brane cannot be identified, and that Eqs. (42), (43) read $\mathcal{H}\Psi = 0$, $\mathcal{H}_i\Psi = 0$, which implies that the wave functional is “timeless”, with no evolution. What matters is that the wave functional on the brane, i.e., at $\mathbf{x} = \mathbf{X}(\sigma)$, has evolution due to the term with $\delta\Psi/\delta T(\sigma)$ on the right hand side of Eq. (42). However, strictly speaking, Eq. (42) is not a true evolution (Schrödinger) equation; it is a set of constraints, valid at any point \mathbf{x} , that can be on the brane or in the bulk. If we perform the Fourier transform, then we obtain the brane analogue of Eq. (27), and the zero mode equation has the form of the Schrödinger equation, with the Hamilton operator $H|_{\mathbf{k}=0} = \int d^3\mathbf{x} \mathcal{H}$. Because of the integration over \mathbf{x} , the quantity $H|_{\mathbf{k}=0}$ has the correct form of a field theoretic Hamiltonian. However, such zero mode Schrödinger equation does not provide a complete description of the system, for which also all higher modes with $\mathbf{k} \neq 0$ are necessary.

4 Discussion and conclusion

We have considered a “point particle” coupled to the gravitational field. The classical constraints become after quantizations a system of equations that comprises the Klein-Gordon, Wheeler-DeWitt and Schrödinger equation. Then we generalize the

theory to p -branes, in which case the Klein-Gordon equation is replaced by the p -brane quantum constraints. In our approach we start from a classical theory in which the point particle or the brane coordinates X^μ and the spatial metric q_{ij} are on the same footing in the sense that they are the quantities that describe the system. In the quantized theory, the system is described by a wave functional $\Psi[X^\mu, q_{ij}]$ that satisfies the system of equations (41)–(43). In the case of a point particle, the latter system becomes (21)–(23). A benefit of such approach is that there is no problem of time. The matter coordinate $X^0 \equiv T$ is time. Moreover, the Wheeler-DeWitt equation has the part $i\partial\Psi/\partial T$, just as the Schrödinger equation.

The wave function(al) $\Psi[X^\mu, q_{ij}]$, satisfying the Klein-Gordon equation, is a generalization of the Klein-Gordon field that depends on X^μ only. In quantum field theory, the Klein-Gordon field, after second quantization, becomes an operator field that, roughly speaking, creates and annihilates particles at spacetime points X^μ . Analogously, we can envisage, that the function(al) $\Psi[X^\mu, q_{ij}]$ should also be considered as a field that can be (secondly) quantized and promoted to an operator that creates or annihilates a particle (in general, a p -brane) at X^μ , together with the gravitational field q_{ij} . An action functional for $\Psi[X^\mu, q_{ij}]$ that leads to Eqs. (21)–(23) should be found and its quantization carried out, together with the calculation of the corresponding vacuum energy density due to the quantum field $\hat{\psi}[X^\mu, q_{ij}]$. It has to be investigated anew, how within such generalized theory vacuum energy influences the gravitational field and what is its effect on the cosmological constant.

Instead of the point particle action (12) that leads to the Klein-Gordon equation, we could take the spinning particle action [19] that leads to the Dirac equation. Then, in the system (21)–(23) we would have the Dirac instead of the Klein-Gordon equation, and $\Psi[X^\mu, q_{ij}]$ would be a generalization of the Dirac field.

We have thus a vision that the quantum field theory of a scalar or spinor field in the presence of a gravitational field could be formulated differently from what we have been accustomed so far. Usually, we have an x^μ -dependent field, e.g., a scalar field $\varphi(x)$ or a spinor field $\psi(x)$, that is a “source” of the gravitational field $g_{\mu\nu}(x)$, decomposed, according to ADM, into $N(x)$, $N^i(x)$ and $q_{ij}(x)$. The action is a functional of those fields, e.g., $I[\varphi(x), N, N^i, q_{ij}(x)]$ or $I[\psi(x), N, N^i, q_{ij}(x)]$, and in the quantized theory we have a wave functional $\Psi[\varphi(x), q_{ij}(x)]$ or $\Psi[\psi(x), q_{ij}(x)]$. In this paper, we investigated an alternative approach, in which the classical action was $I[X^\mu(\tau), N, N^i, q_{ij}(x)]$, and, after quantizing it, we arrived at the wave functional $\Psi[X^\mu, q_{ij}(\mathbf{x})]$, i.e., a generalized field that did not depend on the particle’s position X^μ in spacetime only, but also on the dynamical variables of gravity, $q_{ij}(\mathbf{x})$. Quantum field theory of the generalized field $\Psi[X^\mu, q_{ij}(\mathbf{x})]$ is an alternative to the usual quantum field theoretic approaches to gravity coupled to matter. Since the usual approaches have not yet led us to a consistent theory of quantum gravity, it is worth to investigate what will bring the new approach, conceived in this Letter.

We also considered the case in which, instead of a point particle, $X^\mu(\tau)$, we have a brane $X^\mu(\tau, \sigma^a)$. In a particular case of a spacetime filling brane, we obtained positive or negative cosmological constant, as a consequence of the fact that the brane's momentum p_0 has two discrete values only, namely $\pm\mu_B\sqrt{q}$. In the quantized theory, a state is a superposition of those two possibilities, and the expectation value of the operator \hat{p}_0 is proportional to the effective cosmological constant that can be small or can vanish. The spacetime filling brane could thus be an explanation for a small cosmological constant driving the accelerated expansion of the universe. Finally, there is a possibility that our world is a 3-brane moving in a higher dimensional bulk space. The wave functional, describing such brane, satisfies the Wheeler-DeWitt equation with a Schrödinger like term $i\delta\Psi/\delta T(\sigma)$ that governs the evolution on the brane, whereas there is no evolution in the bulk.

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